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GENERAL SOLUTIONS OF THE DIFFUSION EQUATIONS COUPLED AT BOUNDARY CONDITIONS

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Abstract—New finite integral transform and the corresponding inversion formula are introduced for the solution of the diffusion equations in a finite region of arbitrary geometry and initial conditions with general coupling boundary conditions. The resulting eigenvalue problem does not fall within the range of the conventional Sturm–Liouville system and therefore a new integral condition was devised which serves as an orthogonality relation. The solutions obtained permit the studying of many new problems, such as heat transfer coefficients in concurrent flow double pipe heat exchangers, simultaneous heat and mass transfer in internal gas flows in a duct whose walls are coated with a sublimable material and elsewhere. In addition the Luikov system of equations of a simultaneous mass and heat transfer in a finite capillary porous body of arbitrary geometry are rearranged to the pure diffusion equations coupled only at boundary conditions and consequently to a special case of the problem studied here.

NOMENCLATURE

$B_m(N_2), A_m(N_2)$, boundary coefficient functions defined on S_2;

B_m, A_m, constant boundary coefficients defined on $x = x_0$;

$L_{m,1}, L_{m,2}, L_{m,3}, L_{m,4}, K_{m,1}, K_{m,2}, K_{m,3}, K_{m,4}$, constant boundary coefficients defined on S_1 or $x = x_1$;

$f_m(M), \theta_m(M)$, initial distribution functions in V;

$f_m(x)$, one-dimensional initial distribution functions in x;

$\varphi_m(N_1, \tau), \varphi_m(x_1, \tau)$, source functions on S_1 respectively on $x = x_1$;

$\varphi_m(N_2, \tau), \varphi_m(x_0, \tau)$, source functions on S_2 respectively on $x = x_0$;

$k_m(M)$, prescribed functions defined in V, for which $k_m(N_1)$ are constant;

$k_m(x)$, prescribed function defined in x;

M, point in V;

N_1, N_2, points on S_1 and S_2 respectively;

n, outward normal of S_1 and S_2;

$P_m(M, \tau)$, internal source functions;

$P_m(x, \tau)$, one-dimensional source functions;

S_1 and S_2, boundary of V;

$T_m(M, \tau), T_m^*(M, \tau)$, unsteady potential distributions defined in equations (15) and (25) respectively;

x, co-ordinate;

τ, time variable;

$w_m(M), w_m(x)$, prescribed functions defined in V and x respectively;

$\rho_m(M), \rho_m(x)$, prescribed functions defined in V and x respectively;

$Z_j(M, Fo)$, potentials defined by equations (38);

$\alpha_{m,i}, \beta_{m,j}$, constants defined by equations (45);

ϑ_j, constant defined by equation (40);

μ_i, eigenvalues;
$\psi_{m,i}(M)$, eigenfunctions in M-space;
$\psi_{m,i}(x)$, eigenfunctions in x-space;
div( ), divergence operator in M-space;
grad( ), gradient vector in M-space.

Dimensionless criteria

$Fo = \frac{a_q \tau}{R^2}$ Fourier number;
$Lu = \frac{a_m}{a_q}$, Luikov number;
$Ko = \frac{r c_m(\theta_0 - \theta_*)}{c_q(t_s - t_0)}$, Kossovitch number;
$Pn = \frac{\delta(t_s - t_0)}{c_m(\theta_0 - \theta_*)}$, Posnov number;
ε, phase change criterion, $Ko^* = \varepsilon Ko$;
$\theta_1(M, Fo) = \frac{t - t_0}{t_s - t_0}$, dimensionless temperature;
$\theta_2(M, Fo) = \frac{\theta_0 - \theta}{\theta_0 - \theta_*}$, dimensionless moisture transfer potential;
where t, temperature [°C];
θ, moisture potential [°M];
τ, time [T];
a_q, thermal diffusivity coefficient [L^2T^{-1}];
a_m, diffusion coefficient of moisture in the material [L^2T^{-1}];
r, specific heat of evaporation [L^2T^{-2}];
δ, thermal gradient coefficient [$°C^{-1}$];
c_m, specific isothermal mass capacity of the material [$°M^{-1}$];
c_q, specific heat capacity of the material [$L^2T^{-2}\,°C^{-1}$];
R, characteristic body dimension.

Subscripts
i, 1, 2, 3, 4, ... ;
j, 1 or 2;
m, 1 or 2;
0, initial value;
*, in equilibrium with surrounding air;
s, surrounding.

INTRODUCTION

THE DRYING theory is based on the phenomenon of thermodiffusion found by A. V. Luikov about 30 years ago [1]. It was Luikov who proved experimentally and explained theoretically the moisture migration caused by the temperature gradient [2]. His results have been published in due time in English [3]. Luikov's discovery enabled him on the base of thermodynamics of irreversible processes to define a system of coupled differential equations described in [4]. The majority of the investigations of Luikov's school are reviewed by Fulford [5].

Independently O. Krischer also proposed a system of differential equations for temperature–moisture distribution in a capillary porous body [6, 7]. As we showed in [8] this system is practically identical to that of Luikov. The system defined by de Vries [9] for the case of constant thermophysical parameters [10] is of the Luikov type. Therefore, the results obtained in the present paper captioned with Luikov's nomenclature can be useful to a number of investigators.

It is well known that the coupled system of partial differential equations can be reduced to decoupled of the pure heat conduction, where the potentials are presented by combined variables being a linear combination of output variables. P. S. Henry [11] followed later by J. Krank [12] and M. Smirnov [13] applied this method for the case of monotype boundary conditions. Its application in the general case suggests solutions of the diffusion equations coupled at boundary conditions to be known.

In [14, 15] R. Stein introduced the two-region one-dimensional Sturm–Liouville problem at coupling boundary conditions and found series eigenfunction solutions for heat transfer in concurrent flow, double pipe heat exchanger. E. M. Sparrow and E. C. Spalding [16] developed independently a similar method. Using a new finite integral transform the authors presented a general one-dimensional solution [17] which includes as very special cases the results obtained in [14–16].

The present paper's object is to generalize further this solution for an arbitrary finite geometry, making it applicable for analytical study of unsteady temperature–moisture distribution in capillary-porous bodies.

STATEMENT AND SOLUTION OF THE PROBLEM

In the present study the following equations will be examined

$$w_m(M)\frac{\partial T_m(M,\tau)}{\partial \tau} = \operatorname{div}[k_m(M)\operatorname{grad} T_m(M,\tau)] - \rho_m(M)T_m(M,\tau) + P_m(M,\tau),$$
$$m = 1, 2, M \epsilon V, \tau \geqslant 0 \quad (1)$$

with the initial condition

$$T_m(M,0) = f_m(M) \quad (2)$$

and boundary conditions

$$L_{m,1}T_1(N_1,\tau) + L_{m,2}T_2(N_1,\tau) + L_{m,3}\frac{\partial T_1(N_1,\tau)}{\partial n} + L_{m,4}\frac{\partial T_2(N_1,\tau)}{\partial n} = \varphi_m(N_1,\tau), N_1 \epsilon S_1 \quad (3)$$

$$B_m(N_2)T_m(N_2,\tau) + A_m(N_2)\frac{\partial T_m(N_2,\tau)}{\partial n} = \varphi_m(N_2,\tau), N_2 \epsilon S_2. \quad (4)$$

It is supposed that the solution of the problem can be represented by the expansions

$$T_m(M,\tau) = \sum_{i=1}^{\infty} C_i \psi_{m,i}(M) \quad (5)$$

where $\psi_{m,i}(M)$ are eigenfunctions of the two-region Sturm–Liouville problem

$$\mu_i^2 w_m(M)\psi_{m,i}(M) = \rho_m(M)\psi_{m,i}(M) - \operatorname{div}[k_m(M)\operatorname{grad}\psi_{m,i}(M)] \quad (6)$$

$$L_{m,1}\psi_{1,i}(N_1) + L_{m,2}\psi_{2,i}(N_1) + L_{m,3}\frac{\partial \psi_{1,i}(N_1)}{\partial n} + L_{m,4}\frac{\partial \psi_{2,i}(N_1)}{\partial n} = 0 \quad (7)$$

$$B_m(N_2)\psi_{m,i}(N_2) + A_m(N_2)\frac{\partial \psi_{m,1}(N_2)}{\partial n} = 0 \quad (8)$$

their solution being granted for known.

The governing equations (6) and boundary conditions (7) and (8) lead to an eigenvalue problem having a common set of eigenvalues, but different eigenfunctions. The problem does not belong to the conventional Sturm–Liouville family and therefore, for the determination of the coefficient C_i it is appropriate to derive an integral condition to replace the well-established orthogonality relation.

By conventional manipulations [18], we obtain

$$\int_V w_m(M)\psi_{m,i}(M)\psi_{m,j}(M)\,\mathrm{d}V = \frac{1}{\mu_i^2 - \mu_j^2}\left\{k_m(N_1)\int_{S_1}\begin{vmatrix}\psi_{m,i}(N_1) & \frac{\partial \psi_{m,i}(N_1)}{\partial n}\\ \psi_{m,j}(N_1) & \frac{\partial \psi_{m,j}(N_1)}{\partial n}\end{vmatrix}\mathrm{d}S + \int_{S_2}k_m(N_2)\begin{vmatrix}\psi_{m,i}(N_2) & \frac{\partial \psi_{m,i}(N_1)}{\partial n}\\ \psi_{m,j}(N_2) & \frac{\partial \psi_{m,j}(N_2)}{\partial n}\end{vmatrix}\mathrm{d}S\right\}. \quad (9)$$

From the boundary conditions (8) for $i \neq j$, it follows that

$$\begin{vmatrix}\psi_{m,i}(N_2) & \frac{\partial \psi_{m,i}(N_2)}{\partial n}\\ \psi_{m,j}(N_2) & \frac{\partial \psi_{m,j}(N_2)}{\partial n}\end{vmatrix} = 0. \quad (10)$$

From the conditions (7) $\psi_2(N_1)$ and $\partial \Gamma_2(N_1)n$ are determined as linear combinations of $\psi_1(N_1)$ and $\partial\psi_1(N_1)/\partial n$. A further application of the results yields for

$$\sum_{m=1}^{2}\sigma_m\begin{vmatrix}\psi_{m,i}(N_1) & \dfrac{\partial\psi_{m,i}(N_1)}{\partial n}\\ \psi_{m,j}(N_1) & \dfrac{\partial\psi_{m,j}(N_1)}{\partial n}\end{vmatrix} = 0 \tag{11}$$

where

$$\sigma_1 = \begin{vmatrix}L_{1,3} & L_{1,1}\\ L_{2,3} & L_{2,1}\end{vmatrix}, \qquad \sigma_2 = \begin{vmatrix}L_{1,2} & L_{1,4}\\ L_{2,2} & L_{2,4}\end{vmatrix}. \tag{12}$$

After multiplying the equation (9) for $m = 1$ and $m = 2$ with $k_{3-m}(N_1).\sigma_m$ respectively, adding and integrating into V the results obtained, taking into account that $k_m(N_1)$ are constants, one finds

$$\sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\int_V w_m(M)\psi_{m,i}(M)\psi_{m,j}(M)\,\mathrm{d}V =$$

$$= \frac{1}{\mu_i^2-\mu_j^2}\left\{k_1(N_1)k_2(N_1)\int_{S_1}\sum_{m=1}^{2}\sigma_m\right.$$

$$\times\begin{vmatrix}\psi_{m,i}(N_1) & \dfrac{\partial\psi_{m,i}(N_1)}{\partial n}\\ \psi_{m,j}(N_1) & \dfrac{\partial\psi_{m,j}(N_1)}{\partial n}\end{vmatrix}\mathrm{d}S$$

$$\left.+\sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\int_{S_2}\begin{vmatrix}\psi_{m,i}(N_2) & \dfrac{\partial\psi_{m,i}(N_2)}{\partial n}\\ \psi_{m,j}(N_2) & \dfrac{\partial\psi_{m,j}(N_2)}{\partial n}\end{vmatrix}\mathrm{d}s\right\}. \tag{13}$$

The right-hand side of equation (13) has the same expressions as in relations (10) and (11). Therefore for $j \neq i$ one gets

$$\sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\int_V w_m(M)\psi_{m,i}(M)\psi_{m,j}(M)\mathrm{d}V = 0. \tag{14}$$

Equation (14) will serve as an orthogonality relation. Multiplying equation (6) for $m = 1$ and 2 with $\sigma_m k_{3-m}(N_1)w_m(M)\psi_{m,i}(M)$ respectively, adding and integrating into V the resulting expressions, one finds the expression for C_i. Then (5) may be written as follows

$$T_m(M,\tau) = \sum_{i=1}^{\infty} G_i\psi_{m,i}(M)\tilde{T}_i(\tau) \tag{15}$$

where

$$G_i^{-1} = \sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\int_V w_m(M)\psi_{m,i}^2(M)\,\mathrm{d}V \tag{16}$$

and

$$\tilde{T}_i(\tau) = \sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)$$

$$\times\int_V w_m(M)\psi_{m,i}(M)T(M,\tau)\,\mathrm{d}V. \tag{17}$$

Let us carry out the transition to boundary $\mu_i \to \mu_j$ in equations (13). For (16) the following expression is obtained:

$$G_i = 2\mu_i\left\{k_1(N_1)k_2(N_1)\int_{S_1}\sum_{m=1}^{2}\sigma_m\right.$$

$$\times\begin{vmatrix}\left(\dfrac{\partial\psi_m(N_1)}{\partial\mu}\right)_{\mu=\mu_i} & \left(\dfrac{\partial^2\psi_m(N_1)}{\partial n\partial\mu}\right)_{\mu=\mu_i}\\ \psi_{m,i}(N_1) & \dfrac{\partial\psi_{m,i}(N_1)}{\partial n}\end{vmatrix}\mathrm{d}s$$

$$+\sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\int_{S_2}k_m(N_2)$$

$$\left.\times\begin{vmatrix}\left(\dfrac{\partial\psi_m(N_2)}{\partial\mu}\right)_{\mu=\mu_i} & \left(\dfrac{\partial^2\psi_m(N_2)}{\partial n\partial\mu}\right)_{\mu=\mu_i}\\ \psi_{m,i}(N_2) & \dfrac{\partial\psi_{m,i}(N_2)}{\partial n}\end{vmatrix}\mathrm{d}s\right\}^{-1}. \tag{18}$$

To solve equation (1) at the conditions (2)–(4) the new finite integral transform (17) is to be used. The expressions (15) is sometimes called the inversion formula for (17).

After multiplying the systems (1) and (6) for $m = 1$ and 2 by $\sigma_m k_{3-m}(N_1)\psi_{m,i}(M)$ and $\sigma_m k_{3-m}(N_1)T_m(M,\tau)$ respectively, adding and integrating into V the four results obtained will be

$$\frac{d\bar{T}_i(\tau)}{d\tau} + \mu_i^2 \bar{T}_i(\tau) = k_1(N_1)k_2(N_1)\sum_{m=1}^{2}\sigma_m \times \int_{S_1}\begin{vmatrix}\psi_{m,i}(N_1) & \dfrac{\partial\psi_{m,i}(N_1)}{\partial n}\\ T_m(N_1,\tau) & \dfrac{\partial T_m(N_1\tau)}{\partial n}\end{vmatrix} ds + \sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\left\{\int_{S_2} k_m(N_2) \times \begin{vmatrix}\psi_{m,i}(N_2) & \dfrac{\partial\psi_{m,i}(N_2)}{\partial n}\\ T_m(N_2,\tau) & \dfrac{\partial T_m(N_2,\tau)}{\partial n}\end{vmatrix} ds + \int_V \psi_{m,i}(M)P_m(M,\tau)\,dV\right\}. \quad (19)$$

From (3) and (7), $\psi_2(N_1)$, $\partial\psi_2(N_1)/\partial n$, $T_2(N_1,\tau)$ and $\partial T_2(N_1,\tau)/\partial n$ are determined. The combination and rearrangement of results obtained yields to

$$\sum_{m=1}^{2}\sigma_m \begin{vmatrix}\psi_{m,i}(N_1) & \dfrac{\partial\psi_{mi}(N_1)}{\partial n}\\ T_{m,i}(N_1,\tau) & \dfrac{\partial T_m(N_1,\tau)}{\partial n}\end{vmatrix} = \begin{vmatrix}\varphi_1(N_1,\tau) & L_{1,1}\psi_{1,i}(N_1) + L_{1,3}\dfrac{\partial\psi_{1,i}(N_1)}{\partial n}\\ \varphi_2(N_1,\tau) & L_{2,1}\psi_{1,i}(N_1) + L_{2,3}\dfrac{\partial\psi_{1,i}(N_1)}{\partial n}\end{vmatrix}. \quad (20)$$

From (4) and (8), $A_m(N_2)$ and $B_m(N_2)$ are determined. After summing up the results obtained, one finds

$$\begin{vmatrix}\psi_{m,i}(N_2) & \dfrac{\partial\psi_{m,i}(N_2)}{\partial n}\\ T_m(N_2,\tau) & \dfrac{\partial T_m(N_2,\tau)}{\partial n}\end{vmatrix} = \varphi_m(N_2,\tau)\frac{\psi_{m,i}(N_2) - \dfrac{\partial\psi_{m,i}(N_2)}{\partial n}}{A_m(N_2) + B_m(N_2)}. \quad (21)$$

Substituting (20) and (21) in (19), for $\bar{T}_i(\tau)$ an ordinary first order linear differential equation is obtained, which is easily solved using the transformed initial condition (2) according to (17). The solution is

$$\bar{T}_i(\tau) = e^{-\mu_i^2\tau}\left\{\sum_{m=1}^{2}\sigma_m k_{3-m}(N_1) \times \int_V w_m(M)\psi_{m,i}(M)f_m(M)\,dV + \int_0^{\tau} e^{\mu_i^2\tau}g(\tau)\,d\tau\right\} \quad (22)$$

where

$$g(\tau) = k_1(N_1)k_2(N_1) \times \int_{S_1}\begin{vmatrix}\varphi_1(N_1,\tau) & L_{1,1}\psi_{1,i}(N_1) + L_{1,3}\dfrac{\partial\psi_{1,i}(N_1)}{\partial n}\\ \varphi_2(N_1,\tau) & L_{2,1}\psi_{1,i}(N_1) + L_{2,3}\dfrac{\partial\psi_{1,i}(N_1)}{\partial n}\end{vmatrix} ds + \sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\left\{\int_{S_2} k_m(N_2)\varphi_m(N_2,\tau) \times \frac{\psi_{m,i}(N_2) - [\partial\psi_{m,i}(N_2)/\partial n]}{A_m(N_2) + B_m(N_2)}\,ds + \int_V \psi_{m,i}(M)P_m(M,\tau)\,dV\right\}. \quad (23)$$

Using this solution in the inversion formulae (17) the desired solutions of the problem are obtained.

If $B_m(N_2) = 0$, $L_{1,1} \cdot L_{2,2} - L_{1,2} \cdot L_{2,1} = 0$ and $\rho_m(M) = 0$, then $\mu = 0$ is also eigenvalue of the two-region Sturm–Liouville problem and the corresponding eigenfunctions $\psi_{1,0}$ and $\psi_{2,0}$ must satisfy the equations

$$L_{m,1}\psi_{1,0} + L_{m,2}\psi_{2,0} = 0. \quad (24)$$

Therefore, in this case, additional terms corresponding to the zero-eigenvalue appear in the solutions and it takes the form

$$\begin{aligned} T^*(M,\tau) &= \psi_{m,0}\Big\{\sum_{m=1}^{2}\sigma_m\psi_{m,0}^2 k_{3-m}(N_1)\int_V w_m(M)\,dV\Big\}^{-1} \\ &\times\Big\{\sum_{m=1}^{2}\sigma_m\psi_{m,0}k_{3-m}(N_1)\int_V w_m(M)f_m(M)\,dV \\ &+\int_0^\tau\Big[k_1(N_1)k_2(N_1)\psi_{1,0}\int_{S_1}\begin{vmatrix}\varphi_1(N_1,\tau) & L_{1,1}\\ \varphi_2(N_1,\tau) & L_{2,1}\end{vmatrix}\,ds \\ &+\sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\psi_{m,0}\Big\{\int_{S_2}k_m(N_2)\frac{\varphi_m(N_2,\tau)}{A_m(N_2)}\,ds \\ &+\int_V P_m(M,\tau)\,dV\Big\}\Big]\,d\tau + \sum_{i=1}^{\infty}G_i\psi_{m,i}(M)\bar{T}_i(\tau)\Big\}. \end{aligned} \quad (25)$$

For faster convergence of the series the time integral contained is integrated by parts

$$\int_0^\tau g(\tau)\,e^{\mu_i^2\tau}\,d\tau = \frac{1}{\mu_i^2}\Big\{g(\tau)\,e^{\mu_i^2\tau} - g(0) - \int_0^\tau\frac{\partial g(\tau)}{\partial\tau}\,e^{\mu_i^2\tau}\,d\tau\Big\}. \quad (26)$$

Then the general solution of the problem becomes

$$\begin{aligned} T_m(M,\tau) &= T_m^0(M,\tau) + \sum_{i=1}^{\infty}G_i\psi_{m,i}(M)\,e^{-\mu_i^2\tau} \\ &\times\Big\{\sum_{m=1}^{2}\sigma_m k_{3-m}(N_1)\int_V w_m(M)\psi_{m,i}(M)f_m(M)\,dV \\ &-\frac{1}{\mu_i^2}g(0) - \frac{1}{\mu_i^2}\int_0^\tau\frac{\partial g(\tau)}{\partial\tau}\,e^{\mu_i^2\tau}\,d\tau\Big\} \end{aligned} \quad (27)$$

where

$$T_m^0(M,\tau) = \sum_{i=1}^{\infty}G_i\psi_{m,i}(M)\frac{g(\tau)}{\mu_i^2}. \quad (28)$$

are called pseudo-steady zero-order solutions.

It is easy to show, that the expression (28) is the solution of the differential equation

$$\operatorname{div}[k_m(M)\operatorname{grad}T_m^0(M,\tau)] - \rho_m(M)T_m^0(M,\tau) + P_m(M,\tau) = 0 \quad (29)$$

at boundary conditions

$$L_{m,1}T_1^0(N_1,\tau) + L_{m,2}T_2^0(N_1,\tau) + L_{m,3}\frac{\partial T_1^0(N_1,\tau)}{\partial n} + L_{m,4}\frac{\partial T_2^0(N_1,\tau)}{\partial n} = \varphi_m(N_1,\tau) \quad (30)$$

$$B_m(N_2)T_m^0(N_2,\tau) + A_m(N_2)\frac{\partial T_m^0(N_2,\tau)}{\partial n} = \varphi_m(N_2,\tau). \quad (31)$$

As a matter of fact, after multiplying the systems (29) and (6) for $m = 1$ and 2 by $\sigma_m k_{3-m}(N_1)\psi_{m,i}(M)$ and $\sigma_m k_{3-m}(N_1)T_m^0(M,\tau)$ respectively, adding and integrating into V the four results obtained, as in (20), one gets

$$\bar{T}_i^0(\tau) = \frac{1}{\mu_i^2}g(0). \quad (32)$$

Substituting (32) in the inversion formula (16) we come to the expression (28).

Therefore, the pseudo-steady zero-order solution can be obtained by solving directly, equation (29) at the boundary conditions (30) and (31). The solutions of (29) are particularly convenient when $\varphi_m(N_1,\tau)$, $\varphi_m(N_2,\tau)$, $P_m(M,\tau)$

are not functions of the time. If the latter functions are polynomials or exponentials of the time, the convergence of the series in solutions could be improved in the same way as in [18].

TEMPERATURE AND MOISTURE DISTRIBUTION IN FINITE CAPILLARY POROUS BODY

The transfer processes in a finite capillary porous body of arbitrary geometry is described by Luikov's system [4]

$$\nabla^2\theta_1(M,Fo) = \frac{\partial\theta_1(M,Fo)}{\partial Fo} + Ko^*\frac{\partial\theta_2(M,Fo)}{\partial Fo} \quad (33)$$

$$\nabla^2\theta_2(M,Fo) = Pn\frac{\partial\theta_1(M,Fo)}{\partial Fo} + \left(Ko^*Pn + \frac{1}{Lu}\right)\frac{\partial\theta_2(M,Fo)}{\partial Fo}. \quad (34)$$

The initial potentials are prescribed functions defined in V

$$\theta_m(M,0) = \theta_m(M). \quad (35)$$

The boundary conditions on point N_1 and N_2 on the surface S_1 and S_2 respectively are

$$K_{m,1}\theta_1(N_1,Fo) + K_{m,2}\theta_2(N_1,Fo) + K_{m,3}\frac{\partial\theta_1(N_1,Fo)}{\partial n} + K_{m,4}\frac{\partial\theta_2(N_1,Fo)}{\partial n} = \Omega_m(N_1,Fo) \quad (36)$$

and

$$A(N_2)\frac{\partial\theta_m(N_2,Fo)}{\partial n} + B(N_2)\theta_m(N_2,Fo) = \Omega_m(N_2,Fo). \quad (37)$$

Making the transformation [11–13]

$$Z_j(M,Fo) = \theta_1(M,Fo) + \frac{\vartheta_j^2 - 1}{Pn}\theta_2(M,Fo) \quad (38)$$

the system (33) and (34) can be transformed into the decoupled equations

$$\vartheta_j^2\frac{\partial Z_j(M,Fo)}{\partial Fo} = \nabla^2 Z_j(M,Fo) \quad (39)$$

where

$$\vartheta_j^2 = \frac{1}{2}\left\{1 + K0^*Pn + \frac{1}{Lu} + (-1)^j\sqrt{\left[\left(1 + K0^*Pn + \frac{1}{Lu}\right)^2 - \frac{4}{Lu}\right]}\right\}. \quad (40)$$

From equations (38) it follows

$$\theta_1(M,Fo) = \frac{1}{\vartheta_2^2 - \vartheta_1^2}\{(\vartheta_2^2 - 1)Z_1(M,Fo) - (\vartheta_1^2 - 1)Z_2(M,Fo)\} \quad (41)$$

$$\theta_2(M,Fo) = \frac{1}{\vartheta_2^2 - \vartheta_1^2}\{-Z_1(M,Fo) + Z_2(M,Fo)\}. \quad (42)$$

Substituting (35) into (38) we obtain the initial potentials

$$Z_j(M,0) = \theta_1(M) + \frac{\vartheta_j^2 - 1}{Pn}\theta_2(M). \quad (43)$$

Using (41) and (42) and the fact that $(\vartheta_1^2 - 1)(\vartheta_2^2 - 1) = -Ko^*Pn$ the boundary condition (36) can be rearranged as

$$(\vartheta_2^2 - 1)\left\{\alpha_{m,1}Z_1(N_1,Fo) + \beta_{m,1}\frac{\partial Z_1(N_1,Fo)}{\partial n}\right\} - (\vartheta_1^2 - 1)\left\{\alpha_{m,2}Z_2(N_1,Fo) + \beta_{m,2}\frac{\partial Z_2(N_1,Fo)}{\partial n}\right\} = (\vartheta_2^2 - \vartheta_1^2)\,\Omega_m(N_1,Fo) \quad (44)$$

where

$$\alpha_{mj} = K_{m,1} + \frac{\vartheta_j^2 - 1}{Ko^*}K_{m,2},$$

$$\beta_{mj} = K_{m,3} + \frac{\vartheta_j^2 - 1}{Ko^*}K_{m,4}. \quad (45)$$

If equation (37) for $m = 1$ and the same equation (37) for $m = 2$ multiplied by $(\vartheta_j^2 - 1)/Pn$ are added we obtain the boundary conditions on the surface S_2

$$A(N_2)\frac{\partial Z_j(N_2, Fo)}{\partial n} + B(N_2)Z_j(N_2, Fo) = \Omega_1(N_2, Fo) + \frac{\vartheta_j^2 - 1}{Pn}\Omega_2(N_2, Fo). \quad (46)$$

It is evident, that the solution of the system (39) with the initial conditions (43) and the boundary conditions (44)–(46) is a special case of the general problem studied here. To obtain the desired solution it is sufficient to let: $\tau = Fo$, $T_1(M, \tau) = Z_1(M, Fo)$, $T_2(M, \tau) = Z_2(M, Fo)$, $w_1(M) = \vartheta_1^2$, $w_2(M) = \vartheta_2^2$, $P_m(M, \tau) = 0$, $\rho_m(M) = 0$, $k_m(M) = 1$, $L_{m,1} = (\vartheta_2^2 - 1)\alpha_{m,1}$, $L_{m,2} = -(\vartheta_1^2 - 1)\alpha_{m,2}$, $L_{m,3} = (\vartheta_2^2 - 1)\beta_{m,1}$, $L_{m,4} = -(\vartheta_1^2 - 1)\beta_{m,2}$, $\varphi_m(N_1, \tau) = (\vartheta_2^2 - \vartheta_1^2) \times \Omega_m(N_1, Fo)$, $A_m(N_2) = A(N_2)$, $B_m(N_2) = B(N_2)$, $\varphi_m(N_2, \tau) = \Omega_1(N_2, Fo) +$

$$\frac{\vartheta_m^2 - 1}{Pn}\ \Omega_2(N_2, Fo)$$

ONE-DIMENSIONAL SOLUTIONS

As an application of the general theory let us consider the one-dimensional case studied in [17]:

$$w_m(x)\frac{\partial T_m(x,\tau)}{\partial \tau} = \frac{\partial}{\partial x}\left\{k_m(x)\frac{\partial T_m(x,\tau)}{\partial x}\right\} - \rho_m(x)T_m(x,\tau) + P_m(x,\tau),\ m = 1, 2$$

$$x_0 \leqslant x \leqslant x_1, \tau \geqslant 0 \quad (47)$$

$$T_m(x, 0) = f_m(x) \quad (48)$$

$$A_m\frac{\partial T(x_0,\tau)}{\partial x} + B_m T(x_0, \tau) = \varphi_m(x_0, \tau) \quad (49)$$

$$L_{m,1}T_1(x_1,\tau) + L_{m,2}T_2(x_1,\tau) + L_{m,3}\frac{\partial T_1(x_1,\tau)}{\partial x} + L_{m,4}\frac{\partial T_2(x_1,\tau)}{\partial x} = \varphi_m(x_1,\tau). \quad (50)$$

Equations (6)–(8) giving the eigenvalues and eigenfunctions become

$$\frac{d}{dx}\left\{k_m(x)\frac{d\psi_m(x)}{dx}\right\} + \{\mu^2 w_m(x) - \rho_m(x)\}\psi_m(x) = 0 \quad (51)$$

$$A_m\psi_m'(x_0) + B_m\psi(x_0) = 0 \quad (52)$$

$$L_{m,1}\psi_1(x_1) + L_{m,2}\psi_2(x_1) + L_{m,3}\psi_1'(x_1) + L_{m,4}\psi_2'(x_1) = 0. \quad (53)$$

The solution (27) for the one-dimensional case will be

$$T_m(x,\tau) = T_m^0(x,\tau) + \sum_{i=1}^{\infty} G_i\psi_{m,i}(x)\,e^{-\mu_i^2\tau} \times \left\{\sum_{m=1}^{2}\sigma_m k_{3-m}(x_1)\int_{x_0}^{x_1} w_m(x)\psi_{m,i}(x)f_m(x)\,dx - \frac{1}{\mu_i^2}g(0) - \frac{1}{\mu_i^2}\int_0^{\tau}\frac{\partial g(\tau)}{\partial\tau}e^{\mu_i^2\tau}\,d\tau\right\} \quad (54)$$

where

$$G_i^{-1} = 2\mu_i\left\{k_1(x_1)k_2(x_1)\sum_{m=1}^{2}\sigma_m \times \begin{vmatrix}\left(\frac{\partial\psi_m(x_1)}{\partial\mu}\right)_{\mu=\mu_i} & \left(\frac{\partial^2\psi_m(x_1)}{\partial x\partial\mu}\right)_{\mu=\mu_i} \\ \psi_{m,i}(x_1) & \psi_{m,i}'(x_1)\end{vmatrix} - \sum_{m=1}^{2}\sigma_m k_m(x_0)k_{3-m}(x_1) \times \begin{vmatrix}\left(\frac{\partial\psi_m(x_0)}{\partial\mu}\right)_{\mu=\mu_i} & \left(\frac{\partial^2\psi_m(x_0)}{\partial x\partial\mu}\right)_{\mu=\mu_i} \\ \psi_{m,i}(x_0) & \psi_{m,i}'(x_0)\end{vmatrix}\right\}^{-1} \quad (55)$$

and

$$g(\tau) = k_1(x_1)k_2(x_1) \times \begin{vmatrix}\varphi_1(x_1,\tau) & L_{1,1}\psi_{1,i}(x_1) + L_{1,3}\psi_{1,i}'(x_1) \\ \varphi_2(x_1,\tau) & L_{2,1}\psi_{1,i}(x_1) + L_{2,3}\psi_{1,i}'(x_1)\end{vmatrix}$$

$$- \sum_{m=1}^{2} \sigma_m k_{3-m}(x_1)\Big\{k_m(x_0)\varphi_m(x_0,\tau) \times \frac{\psi_{m,i}(x_0) - \psi'_{m,i}(x_0)}{A_m + B_m} - \int_{x_0}^{x_1} \psi_{m,i}(x)P_m(x,\tau)\,dx\Big\} \quad (56)$$

and the pseudo-steady solution of zero-order $T_m^0(x,\tau)$ can be obtained by solving directly

$$\frac{\partial}{\partial x}\left\{k_m(x)\frac{\partial T_m^0(x,\tau)}{\partial x}\right\} - \rho_m(x)T_m^0(x,\tau) + P_m(x,\tau) = 0 \quad (57)$$

at boundary conditions

$$A_m \frac{\partial T_m^0(x_0,\tau)}{\partial x} + B_m T_m^0(x_0,\tau) = \varphi_m(x_0,\tau) \quad (58)$$

$$L_{m,1}T_1^0(x_1) + L_{m,2}T_2^0(x_1) + L_{m,3}\frac{\partial T_1^0(x_1)}{\partial x} + L_{m,4}\frac{\partial T_2^0(x_1)}{\partial x} = \varphi_m(x_1,\tau) \quad (59)$$

The last terms in equation (55) are not to be found in [17] which could be explained by some author's error.

The solution (25) valid for $B_1 = B_2 = 0$, $L_{1,1}\cdot L_{2,2} - L_{1,2}\cdot L_{2,1} = 0$, $\rho_m(x) = 0$, for the one-dimensional case becomes

$$T_m^*(x,\tau) = \psi_{m,0}\Big\{\sum_{m=1}^{2} \sigma_m \psi_{m,0}^2 k_{3-m}(x_1)\int_{x_0}^{x_1} w_m(x)\,dx\Big\}^{-1} \times \Big\{\sum_{m=1}^{2} \sigma_m\psi_{m,0}k_{3-m}(x_1)\int_{x_0}^{x_1} w_m(x)f_m(x)\,dx + \int_0^{\tau}\Big[k_1(x_1)k_2(x_1)\psi_{1,0}\begin{vmatrix}\varphi_1(x_1,\tau) & L_{1,1}\\ \varphi_2(x_1,\tau) & L_{2,1}\end{vmatrix} - \sum_{m=1}^{2}\sigma_m k_{3-m}(x_1)\varphi_m(x_0,\tau)\frac{\psi_{m,0}}{A_m} + \sum_{m=1}^{2}\sigma_m k_{3-m}(x_1)\psi_{m,0}\int_{x_0}^{x_1} P_m(x,\tau)\,dx\Big]d\tau\Big\} + \sum_{i=1}^{\infty} G_i\psi_{m,i}(x)\bar{T}_i(\tau). \quad (60)$$

The above solutions (54) and (60) permit the easy solving of any particular case of transfer problems in the entrance region of pipes and ducts.

If the substitutions $\rho_m(x) = 0$, $P_m(x,\tau) = 0$, $\varphi_m(x_0,\tau) = 0$, $\varphi_m(x_1,\tau) = 0$, $A_m = 1$, $B_m = 0$, $x_0 = 0$, $x_1 = 1$, $f_m(x) = \text{const}$ are made, from (60) one derives a narrower class of problems to which belong the problems discussed in [14–16].

For the case $f_1 = 0, f_2 = 1, L_{1,1} = 0, L_{1,2} = 0$, $L_{1,3} = K$, $L_{1,4} = 1$, $L_{2,1} = 1$, $L_{2,2} = -1$, $L_{2,3} = K_w$, $L_{2,4} = 0$, $w_1(x) = x^{\Gamma}g_1(x)$ ($\Gamma = 0$ or 1 for duct or pipe respectively), $w_2(x) = x^{\Gamma}g_2(x)$, $T_m(x,\tau) = \xi_m(x,z)$ the solution of [14, 15] is obtained.

For the case $f_1 = 1, f_2 = 1, L_{1,1} = 0, L_{1,2} = 0$, $L_{1,3} = L$, $L_{1,4} = -1$, $L_{2,1} = K$, $L_{2,2} = 1$, $L_{2,3} = 0, L_{2,4} = 0$, $T_1(x,\tau) = \theta(\eta,\chi)$, $T_2(x,\tau) = \varphi(\eta,\chi)$, $k_m(x) = 1$, $w_1(x) = 1$ and $w_2(x) = L$ follows the solution given in [16].

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SOLUTIONS GÉNÉRALES DES EQUATIONS DE DIFFUSION COUPLÉES AUX CONDITIONS AUX LIMITES

Résumé—Une nouvelle transformation intégrale finie et la formule d'inversion correspondante sont introduits pour résoudre les équations de diffusion dans un domaine fini de géométrie arbitraire avec des conditions initiales et aux limites. Le problème résultant ne se situe pas dans le système classique de Sturm–Liouville et par suite une nouvelle condition intégrale a été utilisée comme une relation d'orthogonalité. Les solutions obtenues permettent l'étude de nombreux problèmes nouveaux tels que les coefficients de transfert thermique dans les échangeurs à contre-courant avec deux tubes concentriques, les transferts simultanés de chaleur et de masse pour l'écoulement de gaz à l'intérieur d'un tube dont la paroi est recouverte d'une matière sublimable. En outre le système d'équations de Luikov relatif au transfert simultané de chaleur et de masse dans un matériau microporeux de géométrie arbitraire, a été adapté aux équations de diffusion pure couplées aux conditions aux limites et à un cas spécial du problème étudié ici.

ALLGEMEINE LÖSUNGEN DER MIT RANDBEDINGUNGEN GEKOPPELTEN DIFFUSIONSGLEICHUNGEN

Zusammenfassung—Zur Lösung von Diffusionsgleichungen in einem begrenzten Gebiet beliebiger Geometrie und Anfangsbedingungen mit allgemein gekoppelten Randbedingungen wurden eine theoretisch neue endliche Integraltransformation und die dazugehörigen inversen Gleichungen eingeführt.

Das sich ergebende Eigenwertproblem fällt nicht in das Gebiet des üblichen Sturm–Liouville'sche Systems. Daher wurde eine neue umfassende Bedingung gefunden, die als Orthogonalitäts-Beziehung dient. Die Lösungen gestatten die Behandlung vieler neuer Probleme wie das der Wärmetransport-Koeffizienten in gegenläufigen Zweirohr-Wärmetauschern, des gleichzeitigen Wärme- und Stoffübergangs im Innern von Gasströmen in Rohren, deren Wände mit sublimierenden Stoffen versehen sind u.a.m. Ausserdem wurde das Luikov'sche Gleichungssystem für gleichzeitigen Stoff- und Wärmeübergang in porösen Kapillargefässen mit beliebiger Form, zurückgeführt auf die reinen, nur mit Randbedingungen gekoppelten Diffusionsgleichungen und damit auf eine spezielle Form des hier behandelten Problems.

ОБЩИЕ РЕШЕНИЯ УРАВНЕНИЙ ДИФФУЗИИ, СОПРЯЖЕННЫХ С ПОМОЩЬЮ ГРАНИЧНЫХ УСЛОВИЙ

Аннотация—Приводятся новые методы конечных интегральных преобразований и формулы обратных преобразований, применимые для решения дифференциальных уравнений в конечных областях произвольной геометрии и при начальных условиях с граничными условиями сопряжения общего вида. Так как полученную в результате задачу на собственные значения нельзя свести к задаче Штурма-Лиувилля, разработано новое интегральное условие, являющееся соотношением ортогональности. Полученные решения применимы для широкого круга задач, например, для определения коэффициентов теплообмена в противоточных трубчатых теплообменниках, для расчета процессов совместного тепло- и массообмена при течении газа в каналах со стенками из сублимирующегося материала и в других задачах. Кроме того, система уравнений Лыкова для совместного тепло- и массообмены в конеуном капиллярно-порстом теле произвольной геометрии преобразуется к чисто диффузионным уравнениям, сопяженным только с помощью граничных условий. Частный случай такого решения рассматривается в данной статье.